Math 10A
Worksheet, Discussion #27; Tuesday, 7/24/2018
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1 Second-Order Linear ODEs

1.1 Concepts

1. This is for **linear, homogeneous, constant coefficients, second-order** differential equations. It is of the form

$$\boxed{ay'' + by' + cy = 0.}$$

To solve this, we find the roots of the **characteristic polynomial**: $ar^2 + br + c = 0$. Then depending on the roots, we can determine the **general solution**:

Roots	$r_1 \neq r_2$	r, r	$a \pm bi$
General Solution	$c_1e^{r_1t} + c_2e^{r_2t}$	$c_1e^{rt} + c_2te^{rt}$	$c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$

Once we find the general solution, we can find the solution satisfying our initial conditions by plugging them in and solving for c_1, c_2. An **Initial Value Problem (IVP)** is one where the initial conditions are of the form $y(t_0) = y_0, y'(t_0) = y_0'$. You are told the value of y and y' at the same time t_0. A **Boundary Value Problem (BVP)** is one where the initial conditions are of the form $y(a) = y_a, y(b) = y_b$. You are told the value of y at two different times $t = a, b$. An IVP has a **unique** solution, a BVP may have $0, 1$ **or** ∞ solutions.

1.2 Problems

2. True **FALSE** It is possible for there to be no solutions to an initial value problem.

3. **TRUE** False It is possible for there to be no solutions to a boundary value problem.

4. True **FALSE** It is possible for a BVP to have only 2 solutions.

Solution: A BVP can have 0, 1, or ∞ solutions.

5. Solve the initial value problem given by $3y'' = 15y' - 18y$ and $y(0) = 0$ and $y'(0) = 1$.

Solution: We bring all the y's to one side and get $3y'' - 15y' + 18y = 0$ and our characteristic equation is $3r^2 - 15r + 18 = 3(r^2 - 5r + 6) = 3(r-2)(r-3) = 0$ so $r = 2, 3$. So the solution is of the form $y(t) = c_1 e^{2t} + c_2 e^{3t}$. Plugging in the initial conditions gives $y(0) = c_1 + c_2 = 0$ and $y'(0) = 2c_1 + 3c_2 = 1$ which solving gives $c_2 = 1$ and $c_1 = -1$. Therefore the solution is $y(t) = -e^{2t} + e^{3t}$.

6. Solve the initial value problem $2y'' + 4y' + 2y = 0$ with $y(0) = 0, y'(0) = 1$.

Solution: We guess the solution is of the form $y = e^{rt}$. Plugging this in gives $2r^2 e^{rt} + 4re^{rt} + 2e^{rt} = 2e^{rt}(r^2 + 2r + 1) = 0$ and hence $(r+1)^2 = 0$ so $r = -1$ is a double root. Therefore, the general solution is of the form $y = c_1 e^{-t} + c_2 te^{-t}$. Plugging in our initial conditions gives $y(0) = 0$ or $0 = c_1 e^0 + c_2(0)(e^0) = c_1$ and $y(t) = c_2 te^{-t}$ and $y'(t) = c_2 t(-e^{-t}) + c_2 e^{-t}$ and plugging in $y'(0) = 1$ gives $c_2(0) + c_2(1) = c_2 = 1$ so the solution is $y(t) = te^{-t}$.

7. Solve the boundary value problem of a mass on a spring given by $x'' = -4x$ and $x(0) = 0, x(\pi) = 0$.

Solution: We bring the x's on one side to get $x'' + 4x = 0$ and our characteristic equation is $r^2 + 4 = 0$ with roots $0 \pm 2i$. Therefore, the solution is of the form $x(t) = c_1 e^{0t}\cos(2t) + c_2 e^{0t}\sin(2t) = c_1\cos(2t) + c_2\sin(2t)$. Plugging in the boundary conditions give $c_1\cos(0) + c_2\sin(0) = c_1 = 0$ and $x(\pi) = c_1\cos(2\pi) + c_2\sin(2\pi) = c_1 = 0$ so the solution is of the form $x(t) = c_2\sin(2t)$. Thus, there are infinitely many solutions.

8. Solve the boundary value problem given by $y'' = -y$ and $y(0) = 0, y(\pi) = 1$.

Solution: The differential equation is $y'' + y = 0$ so the characteristic equation is $r^1 + 1 = 0$ or $r = \pm i$. Therefore, the solution is of the form $y(t) = c_1\sin(t) + c_2\cos(t)$. The boundary values give $y(0) = c_2 = 0$ and $y(\pi) = -c_2 = 1$ which cannot happen. Therefore, there are no solutions to this equation.

9. Find the second order linear ODE such that $y(t) = e^{2t}\sin(t)$ is a solution to it.

Solution: Since $e^{2t}\sin(t)$ is a solution, this tells us that the roots are $2 \pm i$. Now in order to find the characteristic equation, we just multiply $(r - (2-i))(r - (2+i)) = r^2 - 4r + 5$. So, the ODE is $y'' - 4y' + 5y = 0$. The initial conditions are $y(0) =, y'(0) = 1$.

10. What is the smallest value of $\alpha > 0$ such that any solution of $y'' + \alpha y' + y = 0$ does not oscillate (does not have any terms of $\sin, \cos$).

Solution: The characteristic equation is given by $r^2 + \alpha r + 1 = 0$. The roots are $\frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}$ and this does not have any terms of $\sin, \cos$ whenever $\alpha^2 - 4 \geq 0$ or when $\alpha^2 \geq 4$. Therefore, we must have $\alpha \geq 2$ and the smallest value is $\alpha = 2$.